THE GENUS TWO GOERITZ GROUP OF $\mathbb{S}^2 \times \mathbb{S}^1$

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ABSTRACT. The genus-g Goeritz group is the group of isotopy classes of orientation-preserving homeomorphisms of a closed orientable 3-manifold that preserve a given genus-g Heegaard splitting of the manifold. In this work, we show that the genus-2 Goeritz group of $\mathbb{S}^2 \times \mathbb{S}^1$ is finitely presented, and give its explicit presentation.

1. INTRODUCTION

Given a genus-g Heegaard splitting of a closed orientable 3-manifold, the genus-g *Goeritz group* is the group of isotopy classes of orientation preserving homeomorphisms of the manifold preserving each of the handlebodies of the splitting setwise. When a manifold admits a unique Heegaard splitting of genus g up to isotopy, we might define the genus-g Goeritz group of the manifold without mentioning a specific splitting. For example, the 3-sphere, $\mathbb{S}^2 \times \mathbb{S}^1$ and lens spaces are known to be such manifolds from [17], [2] and [3].

It is natural to study the structure of those groups and to ask if they are finitely generated or presented, and so finding their generating sets or presentations has been an interesting question. But, the generating sets or the presentations of the groups have been obtained only for few manifolds with their splittings of small genus. It was obtained a finite presentation of the genus-2 Goeritz group for the 3-sphere from the works of [8], [15], [1] and [4], and recently for each lens space $L(p,1)$ in [5]. In addition, finite presentations of the genus-2 Goeritz groups of other lens spaces are given in [7]. Also it was obtained in [10] a finite generating set of the genus-3 Goeritz group of the 3-torus. For the higher genus Georitz groups of the 3-sphere and lens spaces, it is conjectured that they are all finitely presented but it is still known to be an open problem.
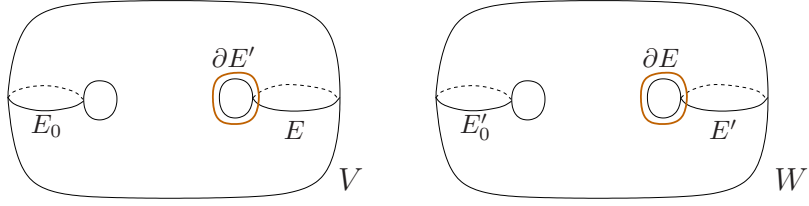
In this work, we show that the genus-2 Goeritz group $\mathcal{G}$ of $\mathbb{S}^2 \times \mathbb{S}^1$ is finitely presented. In the main theorem, Theorem 4.5, their explicit presentations

FIGURE 1.   A genus two Heegaard splitting of $\mathbb{S}^1 \times \mathbb{S}^2$.

are given. We generalize the method developed in [4]. We construct a tree on which the group $\mathcal{G}$ acts such that the quotient of the tree by the action is a single edge, and then find the presentations of the stabilizer subgroups of the edge and each of end vertices.

Throughout the paper, $(V, W; \Sigma)$ will denote the genus-2 Heegaard splitting of $\mathbb{S}^2 \times \mathbb{S}^1$. That is, V and W are genus-2 handlebodies such that $V \cup W = \mathbb{S}^2 \times \mathbb{S}^1$ and $\partial V = \partial W = \Sigma$. All disks in a handlebody are always assumed to be properly embedded and their intersection is transverse and minimal up to isotopy. In particular, if they intersects each other, then the intersection is a collection of pairwise disjoint arcs that are properly embedded in both both disks. Finally, $\mathrm{Nbd}(X)$ will denote a regular neighborhood of X and $\mathrm{cl}(X)$ the closure of X for a subspace X of a polyhedral space, where the ambient space will always be clear from the context.

2. PRIMITIVE DISKS IN A HANDLEBODY

Recall that V is the genus-2 handlebody in $\mathbb{S}^2 \times \mathbb{S}^1$ whose exterior is the handlebody W. A non-separating disk E_0 in V is called a *reducing disk* if there exists a disk E_0' in W such that $\partial E_0 = \partial E_0'$. The disk E_0' is also a reducing disk in W. An essential disk E in V is called *primitive* if there exists an essential disk E' in W such that ∂E meets $\partial E'$ in a single point. Such a disk E' is called a *dual disk* of E, and E' is also primitive in W with its dual disk E. Primitive disks are necessarily non-separating. We remark that $\mathrm{cl}(V - \mathrm{Nbd}(E))$ and $W \cup \mathrm{Nbd}(E)$ (and $V \cup \mathrm{Nbd}(E')$ and $\mathrm{cl}(W - \mathrm{Nbd}(E'))$, respectively) are solid tori, which form a genus-1 Heegaard splitting of $\mathbb{S}^2 \times \mathbb{S}^1$. Moreover, once we have a primitive disk E with its dual disk E', there exist reducing disks E_0 in V and E_0' in W disjoint from $E \cup E'$ and $\partial E_0 = \partial E_0'$. See Figure 1.

The boundary circle of any essential disk in V represents an element of $\pi_1(W)$, a free group of rank two. In particular, if E_0 is a reducing disk in V, then ∂E_0 represents the trivial element of $\pi_1(W)$. For primitive disks, we have the following intepretation, which is a direct consequence of [9].

Lemma 2.1. *Let E be a non-separating disk in V. Then E is primitive if and only if ∂E represents a primitive element of $\pi_1(W)$.*

Let F and G be essential disks in W such that $F \cup G$ cuts off W into a 3-ball. Assign symbols x and y to ∂F and ∂G respectively after fixing

orientations of ∂F and ∂G. Let l be an oriented simple closed curve in ∂W such that l intersects $\partial F \cup \partial G$ transversally and minimally. Then l determines a word in terms of x and y which can be read off from the intersections of l with ∂F and ∂G. Thus l represents an element of the free group $\pi_1(W) = \langle x, y \rangle$. In this set up, the following is a simple criterion for triviality and primitiveness of the elements represented by l, which is found in Lemma 2.2 in [4] with its proof.

Lemma 2.2. *If a word determined by l contains a sub-word of the form yxy^{-1} after a suitable choice of orientations, then l represents a non-trivial, non-primitive element of $\pi_1(W)$.*

The idea of the proof is that, once a word determined by l contains yxy^{-1}, any word determined by l is cyclically reduced and so nonempty, and any cyclically reduced word containing both y and y^{-1} cannot represent a primitive element of $\langle x, y \rangle$.

Let D and E be essential disks in V, and suppose that D intersects E transversely and minimally. Let $C \subset D$ be a disk cut off from D by an outermost arc β of $D \cap E$ in D such that $C \cap E = \beta$. We call such a C an *outermost sub-disk* of D cut off by $D \cap E$. The arc β cuts E into two disks, say H and K. Then we have two essential disks E_1 and E_2 in V which are isotopic to disks $H \cup C$ and $K \cup C$ respectively. We call E_1 and E_2 the *disks from surgery* on E along the outermost sub-disk C of D cut off by $D \cap E$. Observe that each of E_1 and E_2 has fewer arcs of intersection with D than E had, since at least the arc β no longer counts.
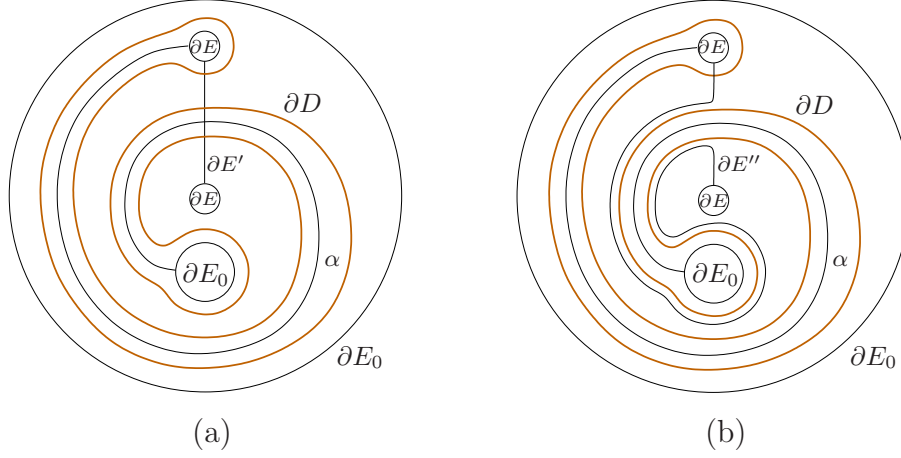
Since E and D are assumed to intersect minimally, E_1 (and E_2) is isotopic to neither E nor D. In particular, if E is non-separating, then the resulting disks E_1 and E_2 are both non-separating and they are not isotopic to each other, and further, E_1 and E_2 are meridian disks of the solid torus V cut off by E, and the boundary circles ∂E_1 and ∂E_2 are not isotopic to each other in the two holed torus ∂V cut off by ∂E.

Lemma 2.3. *Let E_0 be a reducing disk in V disjoint from $E \cup E'$, where E is a primitive disk in V and E' is a dual disk of E. Let D be any non-separating disk in V which is not isotopic to E_0.*

1. *If D is disjoint from E_0, then D is a primitive disk, and hence is not a reducing disk.*
2. *If D intersects E_0, then D is neither a reducing disk nor a primitive disk.*

Proof. Let E_0' be a reducing disk in W such that $\partial E_0 = \partial E_0'$. Then E_0' is also disjoint from $E \cup E'$. Let Σ' be the 4-holed sphere obtained by cutting Σ along ∂E_0 and ∂E. The circle $\partial E_0'$ is identified with one hole coming from ∂E_0, and $\partial E'$ is an arc in Σ' connecting two holes coming from E.

(1) Suppose that D is disjoint from E_0. Consider first the case that D is also disjoint from E. Then D is determined by an arc α properly embedded in Σ' connecting the holes coming from ∂E_0 and ∂E. That is, ∂D is the frontier

FIGURE 2.   The 4-holed sphere Σ'.

of a regular neighborhood of the union of α and the two holes connected by α in Σ'. See Figuire 2 (a). If α intersects $\partial E'$, then take the sub-arc of α which connects $\partial E_0' (= \partial E_0)$ and $\partial E'$ and whose interior is disjoint from $\partial E'$. Then the band sum of E_0' and E' along this sub-arc is a non-separating disk, denoted by E'', in W. We observe that $|\alpha \cap \partial E''| < |\alpha \cap \partial E'|$, and E'' is again a dual disk of E and is disjoint from E_0. See Fig 2 (b). We can repeat this process until we find a dual disk $\widehat{E}'$ of E disjoint from E_0 so that α is disjoint from $\widehat{E}'$. Then ∂D intersects $\partial \widehat{E}'$ in a single point, which implies that D is primitive.

Next, suppose that D intersects E. Let C be any outermost sub-disk of D cut off by $D \cap E$. Then $C \cap \Sigma'$ is an arc properly embedded in Σ' whose end points lie in a single hole coming from ∂E. The arc $C \cap \Sigma'$ is determined by an arc, say β, in Σ' connecting the holes coming from ∂E_0 and ∂E each. Similarly to the case of the arc α, the arc $C \cap \Sigma'$ is the frontier of a regular neighborhood of the union of β and the hole coming from ∂E_0 which β ends in.

If β intersects $\partial E'$, then we repeat the band sum constructions along the sub-arc of β connecting ∂E_0 and $\partial E'$ to find a dual disk $\widehat{E}'$ of E such that $\widehat{E}'$ is disjoint from β and from ∂E_0. Then the arc $C \cap \Sigma'$ is also disjoint from $\widehat{E}'$. One of the disks from surgery on E along C is E_0, and the other one, say E_1, intersects $\widehat{E}'$ in a single point. See Figure 3 (a). That is, E_1 is a primitive disk in V with the dual disk $\widehat{E}'$. The disk E_1 is disjoint from E_0, and further we have $|D \cap E_1| < |D \cap E|$. We repeat the process to find a new primitive disk disjoint from D, and also from E_0, which has a dual disk disjoint from E_0'. Then we go back to the first case.

(2) Suppose that D intersects E_0. Let C_0 be an outermost sub-disk of D cut off by $D \cap E_0$. If C_0 intersects E, the same argument of (1) for the sub-disk
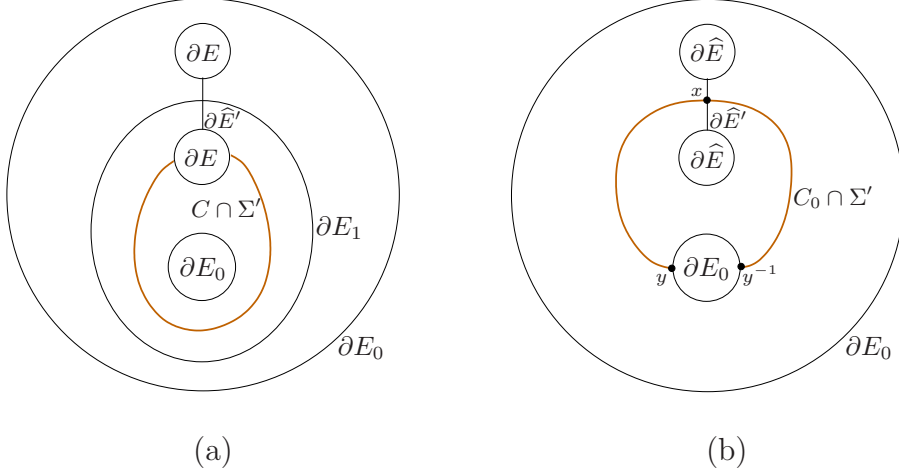
FIGURE 3. (a) The 4-holed sphere Σ'. (b) The 4-holed sphere obtained by cutting Σ along $\partial \widehat{E}$ and ∂E_0.

C_0 (instead of the disk D in (1)) enables us to find a new primitive disk $\widehat{E}$ and its dual disk $\widehat{E}'$ such that $\widehat{E}$ and $\widehat{E}'$ are disjoint from E_0 and E_0', and C_0 is disjoint from $\widehat{E}$ and intersects $\widehat{E}'$ in a single point.

Giving symbols x and y to the oriented circles $\partial \widehat{E}'$ and $\partial E_0'$, respectively, the boundary circle ∂D of D determines a word in terms of x and y. In particular, there exists a word determined by ∂D containing yxy^{-1} which is determined by the sub-arc $C_0 \cap \Sigma'$ (after changing orientations if necessary). See Figure 3 (b). By Lemma 2.2, D is neither a reducing disk nor a primitive disk. $\qquad \square$

In the proof of Lemma 2.3, we see that if a non-separating disk D in V is not isotopic to E_0, then ∂D represents a non-trivial element of $\pi_1(W)$. Thus the reducing disk E_0 is the unique non-separating disk in V such that ∂E_0 represents a trivial element of $\pi_1(W)$ up to isotopy. The following is also a direct consequence of Lemma 2.3.

Lemma 2.4. *There exists a unique non-separating reducing disk E_0 in V. A non-separating disk E in V is primitive if and only if E is not isotopic to and disjoint from E_0.*

Of course, the same result we have for the reducing disk and a primitive disk in W. We call a pair of disjoint, non-isotopic primitive disks in V a *primitive pair* of V, and if a disk in W is a dual disk of each the two disks of the pair, then call it simply a *common dual disk* of the pair.

Lemma 2.5. *Any primitive pair has a common dual disk.*

Proof. Let $\{D, E\}$ be a primitive pair of V, and let E_0 and E_0' be the unique reducing disks in V and W respectively such that $\partial E_0 = \partial E_0'$. Any dual
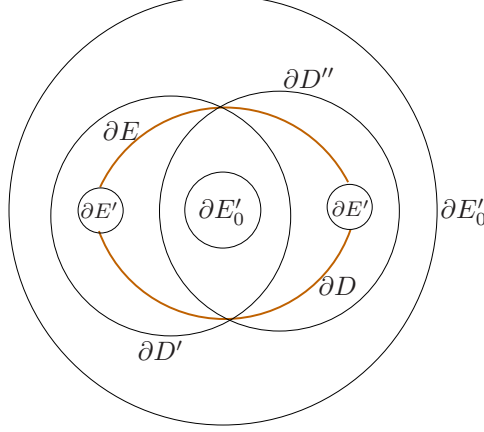
FIGURE 4.   The boundary circles of the two common dual
disks D' and D'' disjoint from E' in the 4-holed sphere Σ''.

disk E' of E is primitive in W, and hence is disjoint from E'_0 by Lemma
2.3. The primitive disk D is disjoint from E_0 and E, thus as in the proof of
Lemma 2.3 (a), we can find a common dual disk $\widehat{E}'$ of D and E by repeating
the band sum constructions.   □

Lemma 2.6. *Let E' be a common dual disk of a primitive pair $\{D, E\}$ of V.
Then there exist exactly two common dual disks, say D' and D'', of $\{D, E\}$
disjoint from E'. Further, D' intersects D'' in a single arc.*

Proof. Let E_0 and E'_0 be the unique reducing disks in V and W, respectively,
such that $\partial E_0 = \partial E'_0$, and let Σ'' be the 4-holed sphere Σ cut off by $\partial E' \cup$
$\partial E'_0$. Then ∂E and ∂D are disjoint arcs properly embedded in Σ' connecting
the two holes coming from $\partial E'$. The boundary circle of any common dual
disk disjoint from E' also lies in Σ'' and intersects each of ∂E and ∂D
in a single point. Thus there exist exactly two such circles $\partial D'$ and $\partial D''$
illustrated in Figure 4, which bound two disks intersecting each other in a
single arc.   □

3. The complex of primitive disks

Let M be an irreducible 3-manifold with compressible boundary. The
disk complex $\mathcal{K}(M)$ of M is a simplicial complex defined as follows. The
vertices of $\mathcal{K}(M)$ are isotopy classes of essential disks in M, and a collection
of $k + 1$ vertices spans a k-simplex if and only if it admits a collection
of representative disks which are pairwise disjoint. Let $\mathcal{D}(M)$ be the full
subcomplex of $\mathcal{K}(M)$ spanned by the vertices of non-separating disks, which
we will call the *non-separating disk complex* of M. It is well known that
$\mathcal{K}(M)$ and $\mathcal{D}(M)$ are both contractible. For example, see Theorems 5.3 and
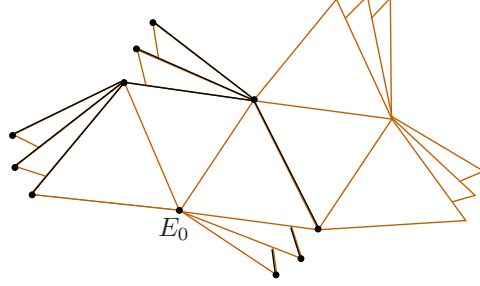5.4 in [12].

FIGURE 5. A portion of $\mathcal{D}(V)$. The primitive disk complex $\mathcal{P}(V)$ is the link of the vertex of the unique reducing disk E_0.

In particular, for a genus-2 handlebody V, the disk complex $\mathcal{K}(V)$ is a 2-dimensional simplicial complex which is not locally finite. Further, we have a precise description of the non-separating disk complex $\mathcal{D}(V)$ of V as follows. First, it is easy to verify that $\mathcal{D}(V)$ is 2-dimensional, and every edge of $\mathcal{D}(V)$ is contained in infinitely but countably many 2-simplices. Next, $\mathcal{D}(V)$ itself is contractible, and also the link of any vertex of $\mathcal{D}(V)$ is a tree each of whose vertices has infinite valency. This is a direct consequence of Theorem 4.2 in [4]. Figure 5 illustrates a portion of $\mathcal{D}(V)$. We note that the entire disk complex $\mathcal{K}(V)$ of V is constructed by attaching infinitely (but countably) many 2-simplices to each edge of $\mathcal{D}(V)$, where each of the new vertices is represented by an essential separating disk in V.

Given a genus-2 Heegaard splitting $(V, W; \Sigma)$ of $\mathbb{S}^2 \times \mathbb{S}^1$, we define the *primitive disk complex* $\mathcal{P}(V)$ to be the full sub-complex of $\mathcal{D}(V)$ spanned by the vertices of primitive disks in V. By Lemma 2.4, $\mathcal{P}(V)$ is the link of the vertex represented by the unique reducing disk E_0 in V (see Figure 5), which implies our key result.

Theorem 3.1. *The primitive disk complex $\mathcal{P}(V)$ is an infinite tree each of whose vertices has infinite valency.*

We note here that in [4], [5], [6], [7] and [11], primitive disk complexes are defined in the same way under various settings, and they are used to obtain presentations of Goeritz groups or their generalizations of given manifolds.

Obviously, the primitive disk complex $\mathcal{P}(W)$ of W is isomorphic to $\mathcal{P}(V)$. Given a primitive disk E of V, define the sub-complex $\mathcal{P}_{\{E\}}(W)$ of $\mathcal{P}(W)$ to be the full sub-complex spanned by the vertices of dual disks of E. Similarly, $\mathcal{P}_{\{D,E\}}(W)$ is the full sub-complex of $\mathcal{P}(W)$ spanned by the vertices of common dual disks of a primitive pair $\{D, E\}$ of V.

Theorem 3.2. *The sub-complexes $\mathcal{P}_{\{E\}}(W)$ and $\mathcal{P}_{\{D,E\}}(W)$ are both infinite trees. In particular, each vertex of $\mathcal{P}_{\{E\}}(W)$ has infinite valency, and each vertex of $\mathcal{P}_{\{D,E\}}(W)$ has valency exactly two.*

Proof. It is clear that $\mathcal{P}_{\{E\}}(W)$ is an infinite sub-complex of the tree $\mathcal{P}(W)$ whose vertices have infinite valency. By Lemmas 2.5 and 2.6, the sub-complex $\mathcal{P}_{\{D,E\}}(W)$ is also an infinite sub-complex of the tree $\mathcal{P}(W)$, and each vertex of $\mathcal{P}_{\{D,E\}}(W)$ has valency two. Thus it remains to show that both of $\mathcal{P}_{\{E\}}(W)$ and $\mathcal{P}_{\{D,E\}}(W)$ are connected.

Let E' and D' be any two dual disks of E. If E' is disjoint from D', then the two vertices represented by them are joined by a single edge. If E' intersects D', then it is easy to see that one of the disks from surgery on E' along an outermost sub-disk of D' cut off by $E' \cap D'$ is the unique reducing disk E_0' in W, and the other one, say E'' is still a dual disk of E. Notice that E'' is disjoint from E' and $|D' \cap E''| < |D' \cap E'|$. By repeating surgery construction, we find a finite sequence of dual disks of E from E' to D', which realizes a path in $\mathcal{P}_{\{E\}}(W)$ joining the two vertices of D' and E'. Thus $\mathcal{P}_{\{E\}}(W)$ is connected. The connectivity of $\mathcal{P}_{\{D,E\}}(W)$ also can be shown in a similar way, by considering surgery on a common dual disk. □

Remark 3.3. It is interesting to compare the sub-complexes $\mathcal{P}_{\{D,E\}}(W)$ of common dual disks for genus-2 Heegaard splittings of the 3-sphere $\mathbb{S}^3$, the lens spaces $L(p,q)$, $p \geq 2$, and $\mathbb{S}^2 \times \mathbb{S}^1$. The followings are known from [4] and [6]. In the case of $\mathbb{S}^3$, there exist infinitely many common dual disks, and each two of them intersect each other. Thus we see that $\mathcal{P}_{\{D,E\}}(W)$ is a collection of infinitely many vertices. For the lens space $L(p,1)$, if $p = 2$, then there exist exactly two common dual disks disjoint from each other. Thus $\mathcal{P}_{\{D,E\}}(W)$ is a single edge. If $p \geq 3$, then there exists a unique common dual disk, and so $\mathcal{P}_{\{D,E\}}(W)$ is a single vertex. For the other lens spaces, $\mathcal{P}_{\{D,E\}}(W)$ is either a vertex or the empty set depending on the choice of $\{D, E\}$, and both exist infinitely many.

4. The genus two Goeritz group of $\mathbb{S}^2 \times \mathbb{S}^1$

In this section, we show that the genus-2 Georitz group $\mathcal{G}$ of $\mathbb{S}^2 \times \mathbb{S}^1$ is finitely presented, and give its explicit presentation. By Theorem 3.1, the primitive disk complex $\mathcal{P}(V)$ is a tree, which is the link of the vertex of the unique reducing disk E_0. Let $\mathcal{T}$ be the barycentric subdivision of $\mathcal{P}(V)$, which is a bipartite tree. As described in Figure 6, the black vertices of $\mathcal{T}$ are the vertices of $\mathcal{P}(V)$, while the white ones are the barycenters of the edges of $\mathcal{P}(V)$. The black vertices correspond to the primitive disks, and the white ones to the primitive pairs in V consisting of the primitive disks representing the two adjoining black vertices.

The group $\mathcal{G}$ acts on the tree $\mathcal{T}$ as a simplicial automorphism, and acts transitively on each of the collections of black vertices and of white vertices. Further, the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge whose one vertex is black and another one white. By the theory of groups acting on trees due to Bass and Serre [16], the group $\mathcal{G}$ can be expressed as the free product of the stabilizer subgroups of two end vertices with amalgamated stabilizer subgroup of the edge.
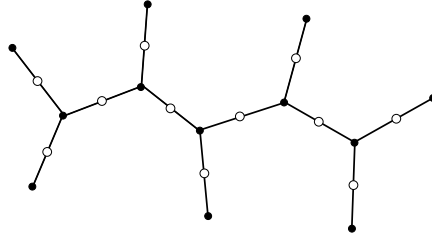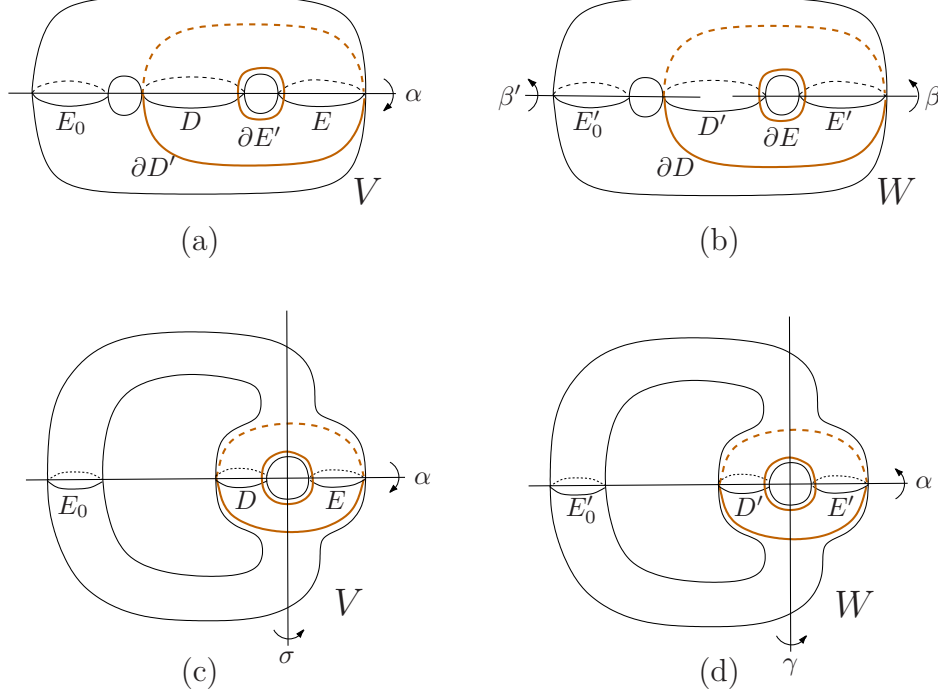
FIGURE 6. A portion of the tree $\mathcal{T}$, which is the barycentric subdivision of $\mathcal{P}(V)$. The black vertices have infinite valency while white ones have valency two.

For convenience, we will not distinguish disks (or pairs of disks) and homeomorphisms from their isotopy classes in their notations. Throughout the section, $\mathcal{G}_{\{A_1, A_2, \cdots, A_k\}}$ will denote the subgroup of $\mathcal{G}$ of elements preserving each of $A_1, A_2, \cdots, A_k$ setwise, where A_i will be (the isotopy classes of) disks or union of disks in V or in W. Then, given a primitive pair $\{D, E\}$ of V, the subgroups $\mathcal{G}_{\{E\}}$, $\mathcal{G}_{\{E \cup D\}}$ and $\mathcal{G}_{\{E, D\}}$ are the stabilizer subgroups of a black vertex, of a white vertex and of the edge of $\mathcal{T}$ joining them respectively. Thus the Goeritz group $\mathcal{G}$ is the free product of $\mathcal{G}_{\{E\}}$ and $\mathcal{G}_{\{E \cup D\}}$ amalgamated by $\mathcal{G}_{\{E, D\}}$.

Fix a primitive pair $\{D, E\}$ of V and a primitive pair $\{D', E'\}$ of common dual disks of D and E. Figures 7 (a) and (b) illustrate those disks with the unique reducing disks E_0 in V and E_0' in W. Notice that $\partial E_0 = \partial E_0'$. We introduce six elements of $\mathcal{G}$, which will turn out to form a generating set of $\mathcal{G}$. Those elements can be described as an orientation preserving homeomorphisms of the surface Σ which can extend to homeomorphisms of the whole $\mathbb{S}^2 \times \mathbb{S}^1$ preserving each of V and W setwise.

In Figure 7, the order two element α of $\mathcal{G}$ is described as an hyperelliptic involution of Σ. The elements β and β' are π-rotations of infinite order such that $\beta\beta' = \alpha$. The order two element γ exchanges D' and E' while fixing each of E and E_0. Notice that D and $\beta(D) = \gamma(D)$ are the two unique common dual disks of the primitive pair $\{D', E'\}$ disjoint from E. In the same manner, the order two element σ exchanges D and E and fixes each of E' and E_0'. Also, D' and $\beta^{-1}(D') = \sigma(D')$ are the two unique common dual disks of the primitive pair $\{D, E\}$ disjoint from E'. Finally, the element ϵ is the Dehn twist about $\partial E_0 = \partial E_0'$. Notice that all of those elements preserve each of E_0 and E_0'.

In addition, we define $\tau = \gamma\beta$ and $\tau' = \tau\sigma$ for later argument. The element τ has infinite order and preserves each of D and E, but sends E' to D'. The element τ' has order two, and exchanges each of D and E, and D' and E' respectively.

FIGURE 7. The elements of the Goeritz group $\mathcal{G}$.

We first find presentations of the three stabilizer subgroups $\mathcal{G}_{\{E\}}$, $\mathcal{G}_{\{E \cup D\}}$ and $\mathcal{G}_{\{E,D\}}$. As mentioned above, the basic idea is to describe the generators of the subgroups as homeomorphisms of the surface Σ preserving the boundary circles of some primitive and semi-primitive disks in V and W.

Lemma 4.1. *The stabilizer subgroup $\mathcal{G}_{\{E\}}$ has the presentation*

$$\langle\, \epsilon \,\rangle \oplus \langle\, \alpha \mid \alpha^2 = 1 \,\rangle \oplus \langle\, \beta, \gamma \mid \gamma^2 = 1 \,\rangle.$$

Proof. By Theorem 3.2, $\mathcal{P}_{\{E\}}(W)$ is the sub-tree of the tree $\mathcal{P}(W)$ spanned by the vertices of the dual disks of E. The barycentric subdivision of $\mathcal{P}_{\{E\}}(W)$, which we denote by $\mathcal{T}_{\{E\}}$, is a bipartite tree, each of whose vertices corresponds to either a dual disk of E or a pair of disjoint dual disks of E. The subgroup $\mathcal{G}_{\{E\}}$ acts on $\mathcal{T}_{\{E\}}$, and its quotient is again a single edge. Thus, fixing a primitive pair $\{D', E'\}$ of dual disks of E as in Figure 7, the subgroup $\mathcal{G}_{\{E\}}$ can be expressed as the free product of stabilizer subgroups $\mathcal{G}_{\{E,E'\}}$ and $\mathcal{G}_{\{E,D' \cup E'\}}$ amalgamated by $\mathcal{G}_{\{E,D',E'\}}$.

First, consider the subgroup $\mathcal{G}_{\{E,E'\}}$. This group also preserves each of E_0 and E_0', and so $\mathcal{G}_{\{E,E'\}} = \mathcal{G}_{\{E,E',E_0,E_0'\}}$. This group is generated by the

elements β, β' and ϵ. Thus we have a presentation of $\mathcal{G}_{\{E,E'\}}$ as $\langle\ \epsilon\ \rangle \oplus \langle\ \beta, \beta' \mid (\beta\beta')^2 = 1,\ \beta\beta' = \beta'\beta\ \rangle$.

Next, the subgroup $\mathcal{G}_{\{E, D' \cup E'\}} = \mathcal{G}_{\{E, D' \cup E', E_0, E'_0\}}$ is generated by α, γ and ϵ, and so it has the presentation $\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle \oplus \langle\ \gamma \mid \gamma^2 = 1\ \rangle$. In a similar way, the index-2 subgroup $\mathcal{G}_{\{E, D', E'\}}$ of $\mathcal{G}_{\{E, D' \cup E'\}}$ has the presentation $\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle$. Observing $\beta\beta' = \alpha$, we have the desired presentation of $\mathcal{G}_{\{E\}}$. $\square$

Lemma 4.2. *The stabilizer subgroup $\mathcal{G}_{\{D \cup E\}}$ has the presentation*

$$\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle \oplus \langle\ \sigma, \tau \mid \sigma^2 = 1,\ (\tau\sigma)^2 = 1\ \rangle.$$

Proof. By Theorem 3.2 again, the full subcomplex $\mathcal{P}_{\{D \cup E\}}(W)$ of $\mathcal{P}(W)$ spanned by the vertices of common dual disks of the pair $\{D, E\}$ is a tree. Denote by $\mathcal{T}_{\{D \cup E\}}$ be the barycentric subdivision of $\mathcal{P}_{\{D \cup E\}}(W)$. Then the quotient of $\mathcal{T}_{\{D \cup E\}}$ by the action of $\mathcal{G}_{\{D \cup E\}}$ is a single edge. Thus, fixing a primitive pair $\{D', E'\}$ of common dual disks of D and E, the subgroup $\mathcal{G}_{\{D \cup E\}}$ can be expressed as the free product of $\mathcal{G}_{\{D \cup E, E'\}}$ and $\mathcal{G}_{\{D \cup E, D' \cup E'\}}$ amalgamated by $\mathcal{G}_{\{D \cup E, D', E'\}}$.

Similarly to the case of $\mathcal{G}_{\{E, D' \cup E'\}}$ in the proof of Lemma 4.1, $\mathcal{G}_{\{D \cup E, E'\}}$ is generated by α, σ and ϵ, and hence has the presentation $\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle \oplus \langle\ \sigma \mid \sigma^2 = 1\ \rangle$. Next, $\mathcal{G}_{\{D \cup E, D', E'\}}$ is a subgroup of $\mathcal{G}_{\{D \cup E, E'\}}$, which does not contain σ since σ does not preserve D'. Thus $\mathcal{G}_{\{D \cup E, D', E'\}}$ has the presentation $\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle$. Finally, recall the order two element $\tau' = \tau\sigma$ of $\mathcal{G}_{\{D \cup E, D' \cup E'\}}$ exchanging D and E, and D' and E' respectively. The subgroup $\mathcal{G}_{\{D \cup E, D' \cup E'\}}$ is the extension of $\mathcal{G}_{\{D \cup E, D', E'\}}$ by τ', so it has the presentation $\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle \oplus \langle\ \tau' \mid \tau'^2 = 1\ \rangle$, and so the desired presentation of $\mathcal{G}_{\{D \cup E\}}$ is obtained. $\square$

Lemma 4.3. *The stabilizer subgroup $\mathcal{G}_{\{D, E\}}$ has the presentation*

$$\langle\ \epsilon\ \rangle \oplus \langle\ \tau\ \rangle \oplus \langle\ \alpha \mid \alpha^2 = 1\ \rangle.$$

Proof. The presentation of $\mathcal{G}_{\{D, E\}}$ is obtained directly from the fact that $\mathcal{G}_{\{D \cup E\}}$ is the extension of $\mathcal{G}_{\{D, E\}}$ by the order-2 element σ exchanging D and E. $\square$

Remark 4.4. The subgroup $\mathcal{G}_{\{D, E\}}$ acts on the tree $\mathcal{T}_{\{D \cup E\}}$. Although this action is transitive on the collections of vertices of common dual disks and of pairs of common dual disks of D and E respectively, the quotient is not a single edge. In fact, one can verify that the quotient is a single loop and can obtain the same presentation of $\mathcal{G}_{\{D, E\}}$ to the above using this loop.

Combining Lemmas 4.1, 4.2 and 4.3 and using $\tau = \gamma\beta$, we obtain the following main result.

Theorem 4.5. *The genus-2 Goeritz group $\mathcal{G}$ has the presentation*

$$\langle\ \epsilon\ \rangle \oplus \langle\ \alpha\ |\ \alpha^2 = 1\ \rangle \oplus \langle\ \beta, \gamma, \sigma\ |\ \gamma^2 = \sigma^2 = (\gamma\beta\sigma)^2 = 1\ \rangle.$$

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